

On the study of a class of non-linear differential equations on compact Riemannian Manifolds.

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Resumo

We study the existence of solutions of the non-linear differential equations on the compact Riemannian manifolds (M^n, g) , $n \geq 2$,

$$\Delta_p u + a(x)u^{p-1} = \lambda f(u, x), \quad (1)$$

where Δ_p is the p -laplacian, with $1 < p < n$. The equation (1) generalizes a equation considered by Aubin [2], where he has considered, a compact Riemannian manifold (M, g) , the differential equation ($p = 2$)

$$\Delta u + a(x)u = \lambda f(u, x), \quad (2)$$

where $a(x)$ is a C^∞ function defined on M and $f(u, x)$ is a C^∞ function defined on $\mathbb{R} \times M$. We show that the equation (1) has solution (λ, u) , where $\lambda \in \mathbb{R}$, $u \geq 0$, $u \not\equiv 0$ is a function $C^{1,\alpha}$, $0 < \alpha < 1$, if $f \in C^\infty$ satisfies some growth and parity conditions.

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1 Introduction

The study of the theory of nonlinear differential equations on Riemannian manifolds has began in 1960 with the so-called Yamabe problem. At a time when little was known about the methods of studying a non-linear equation, the Yamabe problem came to light of a geometric idea and from time sealed a merger of the areas of geometry and differential equations. Let (M, g) be a compact Riemannian manifold of dimension n , $n \geq 3$. Given $\tilde{g} = u^{4/(n-2)}g$ some conformal metrical to the metric g , is well known that the scalar curvatures R and \tilde{R} of the metrics g and \tilde{g} , respectively, satisfy the law of transformation

$$\Delta u + \frac{n-2}{4(n-1)}Ru = \frac{n-2}{4(n-1)}\tilde{R}u^{2^*-1}$$

where Δ denote the Laplacian operator associated to g .

In 1960, Yamabe [17] announced that for every compact Riemannian manifold (M, g) there exist a metric \tilde{g} conformal to g for which \tilde{R} is constant. In another words, this mean that for every compact Riemannian manifold (M, g) there exist $u \in C^\infty(M)$, $u > 0$ on M and $\lambda \in \mathbb{R}$ such that

$$\Delta u + \frac{n-2}{4(n-1)}Ru = \lambda u^{2^*-1}. \quad (Y)$$

In 1968, Trüdinger [16] found an error in the work of Yamabe, which generated a race to solve what became known as the Yamabe problem, today it is completely positively solved, that is, the assertion of

on an invariant (called Yamabe invariant). Then he used tests functions, locally defined, to show that non locally conformal flat manifolds, of dimension $n \geq 6$, satisfied this condition. Finally, for $n \geq 3$ the problem was completely solved by R. Schoen [13].

As previously reported, several disturbances were considered to the Yamabe's problem, all disturbances of analytical character, both in the sense of equation (with the addition of other factors) and in the sense of the operator (the Laplacian for the p -Laplacian), and using the Aubin's idea of estimating corresponding functional. We can cite some articles, such as [6], [7], [9], [10], [11] and [12].

In [15], the author studied the existence of solutions for a class of non-linear differential equation on compact Riemannian manifolds. He establish a lower and upper solutions' method to show the existence of a smooth positive solution for the equation (3)

$$\Delta u + a(x)u = f(x)F(u) + h(x)H(u), \quad (3)$$

where a, f, h are positive smooth functions on M^n , a n -dimensional compact Riemannian manifold, and F, H are non-decreasing smooth functions on \mathbb{R} . In [10] the equation (3) was studied when $F(u) = u^{2^*-1}$ and $H(u) = u^q$ in the Riemannian context, i.e.,

$$\Delta u + a(x)u = f(x)u^{2^*-1} + h(x)u^q, \quad (4)$$

where $0 < q < 1$. In [8] Corrêa, Gonçalves and Melo studied an equation of the type equation (4), in the Euclidean context, with respect to a more general operator than the laplacian operator.

This work, which is organized into four sections, also aims to work with problems related to the equation (Y), although, as we shall see, with different methods from those used by Yamabe, these results were obtained in [14],

In section 2, we enter what we consider as basic concepts necessary to understand it, as some definitions and theorems of embedded.

We consider $F(t, x) = \int_0^t f(s, x)ds$, $B(u) = \int_M F(u(x), x)dV$

and

$$I(u) = \int_M |\nabla u|^p dV + \int_M a|u|^p dV.$$

Given $R > 0$, we also consider $\mathcal{H} = \{u \in H_1^p(M); B(u) = R\}$ and $\mu_R = \inf_{u \in \mathcal{H}} I(u)$.

We proved, in the following theorems

Theorem 1.1. *Given any $R > 0$, the equation (1) has a solution (λ, u) , where $\lambda \in \mathbb{R}$, $u \geq 0$, $u \not\equiv 0$ is a $C^{1,\alpha}$ function, $0 < \alpha < 1$, verifying $B(u) = R$ and $I(u) = \mu_R$, if $f \in C^\infty$ satisfies the following conditions:*

- (p₁) $f(t, x)$ is a strictly increasing odd function on t ;
- (p₂) There exist constants $b > 0$ and $0 < \rho < p^* - 1$ such that $|f(t, x)| \leq b(1 + |t|^\rho)$.

Theorem 1.2. *The equation (1) has a solution (λ, u) , $\lambda \in \mathbb{R}$, $u \in C^{1,\alpha}(M)$ for some $0 < \alpha < 1$, $u \geq 0$ and $u \not\equiv 0$ if $f(t, x)$ satisfies to the following properties:*

- (p₁) $f(t, x)$ is a strictly increasing odd function on t ;
- (p₃) There exist positive constants b and c such that $|f(t, x)| \leq b + c|t|^{p^*-1}$;
- (p₄) $\liminf_{t \rightarrow 0^+} \frac{1}{t^{p^*-1}} [\inf_{x \in M} f(t, x)] = +\infty$.

The function u is strictly positive and increasing for $\lambda \geq 0$.

We list the article by O. Druet [11], where he studied a generalization of (Y) for a more general operator (the p -Laplacian), as the article by Aubin [2], to obtain a solution (λ, u) , $\lambda \in \mathbb{R}$ and $u \in H_1^p$, to the equation (1).

To find such a solution used as a main tool, the Lagrange Multipliers's Theorem, which can be used because of the nature of the equation.

2 Generalization of a nonlinear differential equation

In this section we will work with a generalization of paper of Aubin [2], where he has considered the differential equation (2), namely $\Delta u + a(x)u = \lambda f(u, x)$, on a compact Riemannian manifold (M, g) , where $a(x)$ is a function C^∞ on M and $f(u, x)$ is a C^∞ function on $\mathbb{R} \times M$.

In his paper, Aubin showed that, under certain conditions on $f(u, x)$, the equation (2) has a regular solution whenever $f(u, x)$ satisfies the increasement condition: there are two positive constants b and ρ such that $|f(t, x)| \leq b(1 + |t|^\rho)$, where $0 < \rho \leq (n+2)/(n-2) = 2^* - 1$, $2^* = (2n)/(n-2)$.

We will use the method in [2] to generalize the below equation, in the sense of that the operator will be the p -Laplacian. For this, by the lack of compactness of Sobolev embedded for the critical case (Theorem of compact embedded of Kondrakov) we split the development into two cases: subcritical case ($0 < \rho < p^* - 1$) and the critical case ($\rho = p^* - 1$). This kind of equation was studied by many authors in the Euclidean context. In the Riemannian context we refer mainly to the Druet's article [11] which we extracted regularities' theorems and Maximun principles were used.

Let (M, g) be a compact Riemannian manifold, n -dimensional, $n \geq 3$ and $p \in (1, n)$.

We are interested in the following generalization of the equation (2):

We look for solutions $u \in H_1^p(M) \cap C^0(M)$ and $\lambda \in \mathbb{R}$ for the equation (1), namely

$$\Delta_p u + a(x)u^{p-1} = \lambda f(u, x)$$

where $|f(t, x)| \leq b(1 + |t|^\rho)$, $0 < \rho \leq p^* - 1$, $p^* = pn/(n-p)$ and $\Delta_p u = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p -Laplacian of u .

Remark. If $p = 2$, the equation (1) became to (2), since $\Delta_2 u = \Delta u$.

2.1 Subcritical case

In this section we will study the equation (1) in the subcritical case, i. e., where $0 < \rho < p^* - 1$. The goal is to obtain a solution as the limit of a minimizing sequence for the invariant μ_R that, after using the Dominated Convergence Theorem of Lebesgue, can be directly used in the subcritical case because of the compact embedded of Sobolev, in this case, the convergence to a solution follows easily from Lagrange Multipliers's Theorem.

For the proof of Theorem 1, we need the following lemma:

Lemma 1. *If $f(t, x)$ satisfies the condition (p_1) , then*

- (i) $F(t, x)$ is a non negative and C^∞ function.
- (ii) $F(0, x) = 0$ and $F(\infty, x) = \infty$.
- (iii) $F(t, x)$ is an increasing function for $t \geq 0$.
- (iv) $F(t, x) = F(|t|, x) \forall t$.

Proof of Lemma 1:

- (i) As $F(t, x) = \int_0^t f(s, x)ds$ and f is of C^∞ class, we have that $F \in C^\infty$.

As f is increasing and odd, $f(0, x) = 0$ and if $t \geq 0$, $F(t, x) \geq 0$.

Now, if $t < 0$, take $m > 0$ such that $t = -m$. So

$$\begin{aligned} F(t, x) &= \int_0^t f(s, x)ds = \int_0^{-m} f(s, x)ds \\ &= -\int_{-m}^0 f(s, x)ds = \int_{-m}^0 f(-s, x)ds, \text{ taking } z = -s, \\ &= -\int_m^0 f(z, x)dz = \int_0^m f(z, x)dz \geq 0. \end{aligned}$$

□

$$\begin{aligned}
F(\infty, x) &= \int_0^\infty f(s, x)ds = \int_0^{t_1} f(s, x)ds + \int_{t_1}^\infty f(s, x)ds \\
&\geq A + f(t_1, x) \int_{t_1}^\infty ds = \infty,
\end{aligned}$$

where $A = \int_0^{t_1} f(s, x)ds$. □

(iii) If $0 \leq t_1 < t_2$, then

$$F(t_2, x) = \int_0^{t_2} f(s, x)ds = \int_0^{t_1} f(s, x)ds + \int_{t_1}^{t_2} f(s, x)ds > \int_0^{t_1} f(s, x)ds = F(t_1, x). \quad \square$$

(iv) If $t \geq 0$, $F(t, x) = F(|t|, x)$.

If $t < 0$,

$$\begin{aligned}
F(t, x) &= \int_0^t f(s, x)ds, \text{ taking } s = -z, \\
&= - \int_0^{-t} f(-z, x)dz = \int_0^{-t} f(z, x)dz = F(|t|, x).
\end{aligned}$$

■

Proof of Theorem 1:

By using item (iv) of Lemma 1, we can consider $\mu_R = \inf_{u \in \mathcal{H}_R} I(u)$,

where $\mathcal{H}_R = \{u \in H_1^p(M); u \geq 0 \text{ and } B(u) = R\}$.

Remark By items (ii) and (iii) from Lemma 1, clearly $\mathcal{H}_R \neq \emptyset$.

The proof of the theorem follows in several steps:

Claim 1 There exist $N > 0$ such that, if $u \in \mathcal{H}_R$, then $\|u\|_1 \leq N$.

Firstly fix a $t_o > 0$. Then $\forall u \in \mathcal{H}_R$

$$\begin{aligned}
\|u\|_1 = \int_M u dV &= \int_{\{u < t_o\}} u dV + \int_{\{u \geq t_o\}} u dV \\
&\leq t_o \text{vol}(M) + \int_{\{u \geq t_o\}} u dV.
\end{aligned}$$

For $u \geq t_o > 0$, we have $f(u, x) \geq f(t_o, x) \geq \eta = \inf_{x \in M} f(t_o, x) > 0$ by (p_1) . Whence

$$\begin{aligned}
R = B(u) &= \int_M F(u, x) dV \\
&\geq \int_{\{u \geq t_o\}} F(u, x) dV \\
&= \int_{\{u \geq t_o\}} \left[\int_{t_o}^{u(x)} f(t, x) dt \right] dV \\
&\geq \int_{\{u \geq t_o\}} \left[\int_{t_o}^{u(x)} f(t_o, x) dt \right] dV \\
&\geq \int_{\{u \geq t_o\}} \left[\int_{t_o}^{u(x)} \eta dt \right] dV \\
&= \eta \int_{\{u \geq t_o\}} (u(x) - t_o) dV \\
&= \eta \int_{\{u \geq t_o\}} u(x) dV - \eta t_o \text{vol}(\{u \geq t_o\}) \\
&\geq \eta \int_{\{u \geq t_o\}} u(x) dV - \eta t_o \text{vol}(M).
\end{aligned}$$

where $\{u \geq t_o\} = \{x \in M ; u(x) \geq t_o\}$ and $vol(X)$ is the volume of $X \subseteq M$.

Then,

$$\begin{aligned} \|u\|_1 &= \int_M u(x) dV = \int_{\{u \geq t_o\}} u(x) dV + \int_{\{u < t_o\}} u(x) dV \\ &\leq \frac{R}{\eta} + 2t_o vol(M) = N. \end{aligned}$$

□

Claim 2 μ is finite.

Indeed, by using the below inequality (see [5]), for every $\epsilon > 0$ corresponds a $C(\epsilon) > 0$ such that

$$\int_M |u|^p dV \leq \epsilon \int_M |\nabla u|^p dV + C(\epsilon) \left[\int_M |u| dV \right]^p \quad \forall u \in H_1^p. \quad (5)$$

Therefore, for $u \in \mathcal{H}_R$, we have

$$I(u) = \int_M |\nabla u|^p dV + \int_M a|u|^p dV \geq \int_M |\nabla u|^p dV + \inf_M a \int_M |u|^p dV.$$

If $\inf_M a \geq 0$, we have $I(u) \geq 0$ and, consequently, $\mu \geq 0$.

If $\inf_M a < 0$, by using (5) and Claim 1, we have that

$$\begin{aligned} I(u) &\geq \int_M |\nabla u|^p dV + \epsilon \inf_M a \int_M |\nabla u|^p dV + (\inf_M a) C(\epsilon) \left[\int_M |u| dV \right]^p \\ &\geq (1 + \epsilon \inf_M a) \int_M |\nabla u|^p dV + (\inf_M a) C(\epsilon) N^p \\ &\geq (\inf_M a) C(\epsilon) N^p > -\infty, \end{aligned}$$

Since $\epsilon > 0$ is such that $1 + \epsilon \inf_M a > 0$. What conclude the Claim 2. □

Consider now a sequence $(u_j) \in H_1^p$, $u_j \geq 0$, $B(u_j) = R$ and $I(u_j) \rightarrow \mu_R$ when $j \rightarrow \infty$ (minimizing sequence).

Claim 3 (u_j) is bounded in H_1^p .

Indeed, as $I(u_j) \rightarrow \mu_R$, there exist $K > 0$ such that $|I(u_j)| \leq K \forall j$. Then by (5) and Claim 1, respectively

$$\begin{aligned} \|\nabla u_j\|_p^p &= I(u_j) - \int_M a|u_j|^p dV \leq K + \sup_M |a| \int_M |u_j|^p dV \\ &\leq K + \epsilon \sup_M |a| \|\nabla u_j\|_p^p + C(\epsilon) \sup_M |a| \|u_j\|_1^p \\ &\leq K + \epsilon \sup_M |a| \|\nabla u_j\|_p^p + C(\epsilon) \sup_M |a| N^p. \end{aligned}$$

So

$$(1 - \epsilon \sup_M |a|) \|\nabla u_j\|_p^p \leq K + C(\epsilon) \sup_M |a| N^p.$$

Then, taking $\epsilon > 0$ such that $1 - \epsilon \sup_M |a| > 0$, we obtain

$$\|\nabla u_j\|_p^p \leq C, \quad (6)$$

where $C > 0$ is a positive constant.

Therefore, by (5), (6) and Claim 1, we conclude the proof of Claim 3. □

Now, as H_1^p is reflexive and the Sobolev's embedded $H_1^q \hookrightarrow L^s$ is compact for $1 \leq s < p^*$, the Claim 3 guarantees the existence of a subsequence (u_i) of (u_j) and $u \in H_1^p$ such that

$$u_i \rightharpoonup u \text{ in } H_1^p, \quad (A_1)$$

By (A_1) and (A_2) $I(u) \leq \liminf_{i \rightarrow \infty} I(u_i) = \mu_R$.

By (A_3) $u \geq 0$ a.e. in M . From (A_2) and (p_2) we can use the Lebesgue Dominated Convergence's Theorem (see [5]) to conclude that $B(u) = R$.

Hence $I(u) = \mu_R$, $u \geq 0$ com $u \not\equiv 0$.

So, as B and $I \in C^1(H_1^p)$, taking $S = \{v \in H_1^p; B(v) = R\}$, we have that $B'(v) \neq 0$ for every $v \in S$ and $u \in S$ is such that $I(u) = \inf_{v \in S} I(v)$. Then, by Lagrange Multipliers's Theorem (see [5]), exist $\xi \in \mathbb{R}$ such that $I'(u) = \xi B'(u)$ namely

$$p \int_M |\nabla u|^{p-2} \nabla u \nabla \varphi dV + p \int_M u^{p-1} \varphi dV = \xi \int_M f(u, x) \varphi dV \quad \forall \varphi \in H_1^p.$$

In other words, u is a solution of the equation $\Delta_p u + a(x)u^{p-1} = \lambda f(u, x)$, in the weak sense, where $\lambda = \xi/p$.

Finally, by (p_2) we can use the Regularity Theorem (see [11]) to conclude that exist $0 < \alpha < 1$ such that $u \in C^{1,\alpha}(M)$. ■

Remark If $\lambda \geq 0$, by the Strong maximum principle's Theorem and (see [11]) $u > 0$ in M .

3 Critical case

We will study now the equation (1), where $\rho = p^* - 1$. The problem here is the lack of compactness for Sobolev's embedded when $s = p^*$ (Kondrakov's theorem of embbed) and, to circumvent this difficulty, it will be added an additional condition on $f(u, x)$. The goal is bring down the critical level of f and use Theorem 1.

Proof of Theorem 2:

For each $m \in \mathbb{N}^*$, define

$$f_m(t, x) = \text{signal}(t) \cdot |f(t, x)|^{m/(m+1)}.$$

Then, $f_m(t, x)$ is an odd function and strictly increasing in t and, by (p_3) , it satisfies (p_2) of Theorem 1.

Fixing $R > 0$ (to be clarified further on), as $f(t, x)$ satisfies (p_1) , by items (ii) and (iii) of Lemma 1, exist $\nu \in \mathbb{R}$, $\nu > 0$ such that

$$\int_M F(\nu, x) dV = R, \quad \text{where } F(\nu, x) = \int_0^\nu f(t, x) dt.$$

Now define

$$F_m(t, x) = \int_0^t f_m(s, x) ds$$

and

$$B_m(u) = \int_M F_m(u(x), x) dV.$$

Putting

$$R_m = \int_M F_m(\nu, x) dV,$$

$$\mathcal{H}_m = \{u \in H_1^p(M); u \geq 0 \text{ and } B_m(u) = R_m\}$$

and

$$\mu_m = \inf_{u \in \mathcal{H}_m} I(u),$$

then, by Theorem 1, for each $m \in \mathbb{N}^*$, exist a function $u_m \in C^{1,\alpha}$, $u_m \geq 0$, $u_m \not\equiv 0$ and a real

because $signal(u_m) = 1$. Moreover, u_m performs

$$B_m(u_m) = \int_M F_m(u_m(x), x) dV = R_m$$

and

$$\mu_m = I(u_m).$$

Claim 4 (u_m) is bounded in H_1^p .

Indeed, as

$$F_m(\nu, x) = \int_0^\nu |f(t, x)|^{m/(m+1)} dt \leq \nu + F(\nu, x)$$

we have

$$R_m \leq \nu \cdot vol(M) + R \quad \forall m. \quad (8)$$

On the other hand, fixing $t_o > 0$ and $\eta > 0$ like in proof of Claim 1

$$\begin{aligned} \|u_m\|_1 &= \int_M u_m dV = \int_{\{u_m < t_o\}} u_m dV + \int_{\{u_m \geq t_o\}} u_m dV \\ &\leq t_o vol(M) + \int_{\{u_m \geq t_o\}} u_m dV. \end{aligned}$$

For $u_m \geq t_o > 0$, we have $f(u_m, x) \geq f(t_o, x) \geq \eta > 0$. Whence

$$|f(t_o, x)|^{m/(m+1)} \geq \eta^{m/(m+1)}$$

and

$$\begin{aligned} R_m = B_m(u_m) &= \int_M F_m(u_m, x) dV \\ &\geq \int_{\{u_m \geq t_o\}} F_m(u_m, x) dV \\ &= \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} |f(t, x)|^{m/(m+1)} dt \right] dV \\ &\geq \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} |f(t_o, x)|^{m/(m+1)} dt \right] dV \\ &\geq \int_{\{u_m \geq t_o\}} \left[\int_{t_o}^{u_m(x)} \eta^{m/(m+1)} dt \right] dV \\ &= \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} (u_m(x) - t_o) dV \\ &= \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} u_m(x) dV - \eta^{m/(m+1)} t_o vol(\{u_m \geq t_o\}) \\ &\geq \eta^{m/(m+1)} \int_{\{u_m \geq t_o\}} u_m(x) dV - \eta^{m/(m+1)} t_o vol(M). \end{aligned}$$

Thus, $\int_{\{u_m \geq t_o\}} u_m(x) dV \leq R_m \eta^{-m/(m+1)} + t_o vol(M)$

and, by (8), exist a $C > 0$ such that

$$\|u_m\|_1 \leq C, \quad \forall m. \quad (9)$$

we obtain

$$\begin{aligned}
\|\nabla u_m\|_p^p &= I(u_m) - \int_M a|u_m|^p dV \\
&\leq \nu^p \int_M a(x) dV + \sup_M |a| \int_M |u_m|^p dV \\
&\leq \nu^p \int_M a(x) dV + \epsilon \sup_M |a| \|\nabla u_m\|_p^p + C(\epsilon) \sup_M |a| \|u_m\|_1^p,
\end{aligned}$$

where $\epsilon > 0$ and $C(\epsilon) > 0$ came from (5).

By taking $\epsilon > 0$ small enough so that $1 - \epsilon \sup_M |a| > 0$ we have, by (9), that exist $C > 0$ such that

$$\|\nabla u_m\|_p^p \leq C \quad \forall m. \quad (10)$$

Finally, by using (5), (9) and (10) we conclude the Claim 4. \square

Claim 5 (λ_m) is bounded in \mathbb{R} .

Indeed, multiplying (7) by u_m and integrating on M , we obtain

$$I(u_m) = \lambda_m \int_M u_m |f(u_m, x)|^{m/(m+1)} dV. \quad (11)$$

On the other hand, as $\|u_m\|_{H_1^p} \leq C$, there is $A > 0$ such that

$$|I(u_m)| \leq A \quad \forall m. \quad (12)$$

By (p_1) we have

$$\begin{aligned}
R_m &= \int_M F_m(u_m, x) dV = \int_M \left[\int_0^{u_m} |f(t, x)|^{m/(m+1)} dt \right] dV \\
&\leq \int_M \left[\int_0^{u_m} |f(u_m, x)|^{m/(m+1)} dt \right] dV \\
&= \int_M u_m |f(u_m, x)|^{m/(m+1)} dV.
\end{aligned}$$

Now, by using (11), (12) and the above expression, we obtain

$$A \geq |I(u_m)| = |\lambda_m| \int_M u_m |f(u_m, x)|^{m/(m+1)} dV \geq |\lambda_m| R_m.$$

Namely,

$$|\lambda_m| R_m \leq A \quad \forall m. \quad (13)$$

Furthermore, when $m \rightarrow \infty$, $f^{m/(m+1)}(t, x) \rightarrow f(t, x)$ and the convergence is dominated by $1 + f(t, x)$, it is integrable over $[0, \nu]$, whence $F_m(\nu, x) \rightarrow F(\nu, x)$. And the convergence is dominated by $\nu + F(\nu, x)$. Then we have (see [5])

$$R_m \rightarrow R \quad \text{when } m \rightarrow \infty.$$

As $R > 0$, we can assume that there is $C_o > 0$ such that $R_m > C_o \quad \forall m$.

So, by (13), $|\lambda_m| \leq \frac{A}{C_o}$, give us the proof of Claim 5. \square

As H_1^p is reflexive the Sobolev's embedded $H_1^p \hookrightarrow L^s$ is compact for $1 \leq s < p^*$, from Claims 4 and 5, there are (u_m) subsequence of (u_m) , (λ_m) subsequence of (λ_m) , $u \in H_1^p$ and $\lambda \in \mathbb{R}$ such that

$$u_m \longrightarrow u \text{ a.e. in } M \text{ and} \quad (B_3)$$

$$\lambda_m \longrightarrow \lambda. \quad (B_4)$$

Remark We are using in the proofs the same notation to denote a subsequence.

With this, $u \geq 0$ and $|f(u_m, x)|^{m/(m+1)} \longrightarrow f(u, x)$ a.e. in M .

Claim 6 $|f(u_m, x)|^{m/(m+1)}$ is bounded in $L^{p^*/(p^*-1)}$.

Indeed, by Hölder's inequality

$$\begin{aligned} \| |f(u_m, x)|^{m/(m+1)} \|_{p^*/(p^*-1)}^{(m+1)/m} &= \| f(u_m, x) \|_{[m/(m+1)][p^*/(p^*-1)]} \\ &\leq \text{vol}(M)^{(p^*-1)/(m+1)p^*} \| f(u_m, x) \|_{p^*/(p^*-1)} \\ &\leq C \| f(u_m, x) \|_{p^*/(p^*-1)} \end{aligned}$$

and by (p_3)

$$\begin{aligned} \| f(u_m, x) \|_{p^*/(p^*-1)} &= \left[\int_M |f(u_m, x)|^{p^*/(p^*-1)} dV \right]^{(p^*-1)/p^*} \\ &\leq \left[\int_M (b_1 + c_1 |u_m|^{p^*}) dV \right]^{(p^*-1)/p^*} \\ &\leq C + C \| u_m \|_{p^*}^{p^*-1} \leq C \end{aligned}$$

this last inequality is due to Claim 4 and $H_1^p \hookrightarrow L^{p^*}$. b_1, c_1 are positive constants and C represent several positive constants, not necessarily the same.

We conclude the proof of Claim 6. \square

Consequently, (see [4]), considering a subsequence,

$$|f(u_m, x)|^{m/(m+1)} \rightharpoonup f(u, x) \text{ em } L^{p^*/(p^*-1)}. \quad (14)$$

Analogously, by Claim 4, $|\nabla u_m|^{p-2} \nabla u_m$ is bounded in $L^{p/(p-1)}$. Then, considering a subsequence $|\nabla u_m|^{p-2} \nabla u_m \rightharpoonup \Sigma$ in $L^{p/(p-1)}$, for some $\Sigma \in L^{p/(p-1)}$.

Now, by using (7), (p_3) , (B_2) and (B_4) we conclude that $\text{div}(|\nabla u_m|^{p-2} \nabla u_m)$ is bounded in L^1 , we have that $\Sigma = |\nabla u|^{p-2} \nabla u$ (see [11]). Therefore,

$$|\nabla u_m|^{p-2} \nabla u_m \rightharpoonup |\nabla u|^{p-2} \nabla u \text{ in } L^{p/(p-1)}. \quad (15)$$

To conclude the proof of the Theorem 2, we remember from (7) that

$$\int_M |\nabla u_m|^{p-2} \nabla u_m \nabla \varphi dV + \int_M a(u_m)^{p-1} \varphi dV = \lambda_m \int_M |f(u_m, x)|^{m/(m+1)} \varphi dV, \quad \forall \varphi \in H_1^p.$$

Taking $m \rightarrow \infty$, and using (B_2) , (B_4) , (14) and (15), we obtain

$$\int_M |\nabla u|^{p-2} \nabla u \nabla \varphi dV + \int_M a u^{p-1} \varphi dV = \lambda \int_M f(u, x) \varphi dV, \quad \forall \varphi \in H_1^p.$$

Namely, u is a solution (in the weak sense) of the equation (1).

To regularize the solution we use the hypothesis (p_3) (see [11]). With this, there is some $0 < \alpha < 1$ such that $u \in C^{1,\alpha}(M)$.

As we already know that $u \geq 0$, to finish the proof of the theorem we have to show that $u \not\equiv 0$.

By (B_1) and (B_2) , we have that

$$I(u) \leq \liminf_{m \rightarrow \infty} I(u_m). \quad (16)$$

For some function $u_o \in H_1^p$, $u_o \geq 0$, $u_o \not\equiv 0$, if we have $I(u_o) \leq 0$, then for each m , there is $k_m > 0$ such that $B(k_m u_o) = R_m$ and $I(k_m u_o) = (k_m)^p I(u_o) \leq 0$ (see Lemma 1). Then

But, if $I(u) < 0$, we have that $u \not\equiv 0$, this also prove the theorem.

Let us prove, then, the case where $I(u_m) > 0$ for all $m \geq 1$.

By (p_3) , we have

$$|f(t, x)|^{m/(m+1)} \leq b_1 + c_1 |t|^{[m/(m+1)][p^*-1]} \leq b_1 + c_1 + c_1 |t|^{p^*-1}$$

where b_1 and c_1 are positive constants. Thus, considering $b_2 = b_1 + c_1$, we obtain

$$R_m = \int_M \left[\int_0^{u_m} |f(t, x)|^{m/(m+1)} dt \right] dV \leq b_2 \|u_m\|_1 + \frac{c_1}{p^*} \|u_m\|_{p^*}^{p^*}. \quad (17)$$

As $H_1^p \hookrightarrow L^{p^*}$, there is K and $D > 0$ such that

$$\|\varphi\|_{p^*}^p \leq K \|\nabla \varphi\|_p^p + D \|\varphi\|_p^p \quad \forall \varphi \in H_1^p.$$

From this fact

$$\|\varphi\|_{p^*}^{p^*} \leq [K \|\nabla \varphi\|_p^p + D \|\varphi\|_p^p]^{p^*/p} \quad \forall \varphi \in H_1^p. \quad (18)$$

Then, by (17) and (18) we have

$$R_m - b_2 \|u_m\|_1 \leq \frac{c_1}{p^*} [K \|\nabla u_m\|_p^p + D \|u_m\|_p^p]^{p^*/p}. \quad (19)$$

If $R_m - b_2 \|u_m\|_1 < 0$, then $\|u_m\|_1 > \frac{R_m}{b_2}$, what give us, by (B_2) and by $R_m \rightarrow R > 0$, that $\|u\|_1 \geq \frac{R}{b_2} > 0$, in other words, $u \not\equiv 0$.

Now, if $R_m - b_2 \|u_m\|_1 \geq 0$, we have $1 - \frac{b_2}{R_m} \|u_m\|_1 \geq 0$ and by (19) we obtain

$$\begin{aligned} \left(\frac{R_m p^*}{c_1} \right)^{p/p^*} \left(1 - \frac{b_2}{R_m} \|u_m\|_1 \right) &\leq \left(\frac{R_m p^*}{c_1} \right)^{p/p^*} \left(1 - \frac{b_2}{R_m} \|u_m\|_1 \right)^{p/p^*} \\ &\leq K \left[I(u_m) - \int_M a |u_m|^p dV \right] + D \|u_m\|_p^p \\ &\leq \mu_m K + D_o \|u_m\|_p^p \end{aligned} \quad (20)$$

where $D_o > 0$.

Claim 7 There is $\epsilon > 0$ such that for all $m \geq 1$ and a convenient $R > 0$

$$K \mu_m < \left(\frac{R p^*}{c_1} \right)^{p/p^*} - 2\epsilon.$$

Indeed, by (p_4) there is a sequence of real numbers $\nu_i > 0$ such that $\nu_i \rightarrow 0$, when $i \rightarrow \infty$, and $f(t, x) > i t^{p^*-1}$ for all $t \in (0, \nu_i)$. This implies that

$$F(\nu_i, x) = \int_0^{\nu_i} f(t, x) dt > \frac{i}{p^*} (\nu_i)^{p^*}$$

and, consequently,

$$R_i = \int_M F(\nu_i, x) dV > \frac{i}{p^*} (\nu_i)^{p^*} \text{vol}(M).$$

Taking

$$(R_m)_i = \int_M F_m(\nu_i, x) dV,$$

$$(\mathcal{H}_m)_i = \{u \in H_1^p(M); u \geq 0 \text{ and } B_m(u) = (R_m)_i\}$$

we obtain

$$(\mu_m)_i \leq I(\nu_i) = (\nu_i)^p \int_M a(x) dV.$$

With this

$$\frac{(\mu_m)_i}{(R_i)^{p/p^*}} \leq \left[(\nu_i)^p \int_M a(x) dV \right] / \left[\left(\frac{i}{p^*} \right)^{p/p^*} (\nu_i)^p \cdot \text{vol}(M)^{p/p^*} \right] \longrightarrow 0 \text{ when } i \rightarrow \infty.$$

Remark Remember that $R_i > 0$ and we are considering the case where $(\mu_m)_i > 0$ for all m and $i \geq 1$.

Hence,

$$\frac{K(\mu_m)_i}{[(R_i p^*)/c_1]^{p/p^*}} \longrightarrow 0 \text{ when } i \rightarrow \infty, \forall m \geq 1.$$

Then, for a big enough i , taking $R = R_i$ and $\mu_m = (\mu_m)_i$, we have that there is $\epsilon_o > 0$ such that, for all $m \geq 1$

$$\frac{K(\mu_m)}{[(R p^*)/c_1]^{p/p^*}} < 1 - \epsilon_o$$

and taking $2\epsilon = \epsilon_o [(R p^*)/c_1]^{p/p^*}$ we conclude proof of Claim 7. \square

Now, by Claim 7 and the fact that $R_m \rightarrow R$ when $m \rightarrow \infty$, after some m_o

$$K\mu_m + \epsilon < \left(\frac{R_m p^*}{c_1} \right)^{p/p^*}$$

and, by using (20), we obtain

$$(K\mu_m + \epsilon) \left(1 - \frac{b_2}{R_m} \|u_m\|_1 \right) \leq K\mu_m + D_o \|u_m\|_p^p.$$

Then,

$$\epsilon - (K\mu_m + \epsilon) \frac{b_2}{R_m} \|u_m\|_1 \leq D_o \|u_m\|_p^p$$

consequently

$$\epsilon \leq (K\mu_m + \epsilon) \frac{b_2}{R_m} \|u_m\|_1 + D_o \|u_m\|_p^p \quad (21)$$

and since that $\mu_m > 0$, $R_m \rightarrow R$ when $m \rightarrow \infty$, by (B₂) and (21) $u \not\equiv 0$.

Finally, if $\lambda \geq 0$, by Strong maximum principle (see [11]), $u > 0$. \blacksquare

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